A DYNAMIC LEONTIEF MODEL WITH NON-RENEWABLE RESOURCES

Adél Floriska, Imre Dobos

Abstract
In this paper we study a generalisation of the dynamic Leontief input-output model. We extend the standard dynamic Leontief model with the balance equation of non-renewable resources. Obviously the non-renewable stocks will decrease exploiting primary resources. In this study we examine the controllability of this extended model taking the consumption as control parameter. Supposing a balanced growth both for consumption and production whether how long will cover these scare resources the inputs of production and how the lifetime of the system depends on the balanced growth rate and on the consumption as well? To this investigations we use the classic results of the control theory and the eigenvalue problems of the linear algebra.

Keywords:
Leontief model, dynamical systems, eigenvalue problem, control theory, non-renewable resources, environmental management

1. Introduction

Solution of environmental problems such as exhaustible natural resources urges a searching analysis of the interaction between the development of the economy and the natural environment. In this paper we present an empirical environmental impact study focusing on the interdependence of the economic activities and the associated flow of non-renewable raw materials. Despite known simplifications input-output analysis is a useful instrument to study strategic directions of economic development, considering besides dynamic intersectoral links also the interaction between the sectors of economy and the natural environment. Such analysis is important for the development of the state regulation programs, which are integral part of environmental-economic policy for any developed market system.

In this regard we consider the well-known dynamic Leontief input-output model augmented with the balance equation of exhaustible primary resources. One of the questions arising in this ecological input-output model, whether is this dynamic model controllable considering the consumption as control parameter? In other words can the exploitation of the natural resources be influenced through controlling the consumption in the economy? As it is known a growing consumption requires growing production as well, therefore the non-renewable stocks will decrease. The other question we are intend to answer whether how long will cover these scare resources the inputs of production supposing a balanced growth both for consumption and production as well? How the lifetime of the system depends on the balanced growth rate and on the consumption as well?

To this investigations we use the classic results of the control theory and the eigenvalue problems of the linear algebra.The next section presents the basic equation system of the augmented dynamic Leontief system. The following sections examine the controllability and the balanced growth path of the system respectively.

2. The augmented dynamic Leontief model

Our model is based on the equations of the dynamic multi-sector input-output model well known in the literature. We extend this model with the balance equation of exhaustible resources in order to analyse the linkage between the economy and the environment.

Suppose that there are n economic industries each industry producing a single economic commodity and m primary resources used by the sectors of productions. (Each industry may use more resources but at most m and let us suppose that $m<n$.) The input-output balance of the entire economy can be described by the equations of economic goods and resources. The equation of goods describes the balance between the total output of goods of production and the sum of total input of goods of all activities of the economy and the consumed goods.

$$x_t = Ax_t + B(x_{t+1} - x_t) + c_t \qquad (1)$$

The equation of resources describes the relation between the stock of exhaustible resources remained after the exploitation of its in a given year and the stock of resources disposable in the next year.

$$R_{t+1} = R_t - Dx_t \qquad (2)$$

Where

- x_t is the n-dimensional vector of gross industrial outputs,
- c_t is the n-dimensional vector of final consumption demands for economic commodities,
- A is the $(n\times n)$ matrix of conventional input coefficients, showing the input of goods that are required to produce a unit of product,
- B is the $(n\times n)$ matrix of capital coefficients, showing the invested products to increase the output of the producing sectors by a unit,
- D is the $(m\times n)$ matrix of input coefficients of resources, showing the input of resources for industries that are required to produce a unit of product,
- R_t is the m-dimensional vector of non-renewable resources.

Assumptions

Throughout the paper it is assumed that the matrices A, B and D are nonnegative, B is nonsingular and c_t is a nonnegative vector.

Then the above two equations can be written in explicit vector form:

$$\begin{bmatrix} x_{t+1} \\ R_{t+1} \end{bmatrix} = \begin{bmatrix} I + B^{-1}(I - A) & 0 \\ -D & I \end{bmatrix} \begin{bmatrix} x_t \\ R_t \end{bmatrix} - \begin{bmatrix} B^{-1} \\ 0 \end{bmatrix} c_t$$

where I is the $(n\times n)$ identity matrix.

For further analysis it is more convenient to write the above discrete-time linear system in the well-known matrix-vector form as:

$$\mathrm{q}_{t+1} = \mathrm{A}_q \mathrm{q}_t + \mathrm{B}_q \mathrm{u}_t \qquad (3)$$

Where $A_q = \begin{bmatrix} I + B^{-1}(I-A) & 0 \\ -D & I \end{bmatrix}$ is the $(m+n)\times(m+n)$ state matrix, B_q = - $\begin{bmatrix} B^{-1} \\ 0 \end{bmatrix}$ is $(m+n)\times n$ effect matrix of the control, $q_t = \begin{bmatrix} x_t \\ R_t \end{bmatrix}$ is the $m+n$ - dimensional state vector and $u_t = c_t$ is the n - dimensional control vector.

In the next section we examine the controllability of system (3).

3. Controllabillity of the model

The controllability of a system means the property of being able to steer the system from any given initial state to any other state in a finite time period by means of a suitable choice of control functions (Elaydi1996). Applying Kalman's theorem (Kalman 1960, see also Elaydi 1996) we get that system (3) is controllable if and only if the rank condition

$$rank\lfloor B_q \quad A_q B_q \quad \dots \quad A_q^{m+n-1} B_q \rfloor = m+n$$

is satisfied.

Lemma 1.
The system (3) is controllable if and only if $rank\, D = m$.

Proof. By using the property of matrices related to theirs rank it is easy to see that the controllability of system (3) is equivalent to the controllability of the system $q_{t+1} = (A_q - I)q_t + B_q u_t$

Define the $n\times nm$ controllability matrix $W := \lfloor B_q \quad (A_q - I)B_q \quad \dots \quad (A_q - I)^{m+n-1} B_q \rfloor =$

$$\left[\begin{bmatrix} B^{-1} \\ 0 \end{bmatrix}, \begin{bmatrix} B^{-1}(I-A)B^{-1} \\ -DB^{-1} \end{bmatrix}, \begin{bmatrix} (B^{-1}(I-A))^2 B^{-1} \\ -DB^{-1}(I-A)B^{-1} \end{bmatrix}, \dots, \begin{bmatrix} (B^{-1}(I-A))^{m+n-1} B^{-1} \\ -D(B^{-1}(I-A))^{m+n-2} B^{-1} \end{bmatrix}\right]. \tag{4}$$

The controllability condition that W has rank $m+n$ means that the rows of the matrix W are linearly independent. We shall show that the matrix W has exactly $m+n$ linearly independent rows if and only if $rank\, D = m$.

Let y_1 be an n and y_2 an m dimensional vector of constants. Matrix W has exactly $m+n$ linearly independent rows if whenever $[y_1, y_2]W = 0$, then we must have $y_1 = 0$ and $y_2 = 0$. By applying the notation (4) on equation $[y_1, y_2]W = 0$ we obtain

$$[y_1, y_2]\left[\begin{bmatrix} B^{-1} \\ 0 \end{bmatrix}, \begin{bmatrix} B^{-1}(I-A)B^{-1} \\ -DB^{-1} \end{bmatrix}, \begin{bmatrix} (B^{-1}(I-A))^2 B^{-1} \\ -DB^{-1}(I-A)B^{-1} \end{bmatrix}, \dots, \begin{bmatrix} (B^{-1}(I-A))^{m+n-1} B^{-1} \\ -D(B^{-1}(I-A))^{m+n-2} B^{-1} \end{bmatrix}\right] = 0. \tag{5}$$

As *rank* B^{-1}= *n* therefore the equation $[y_1, y_2]W = 0$ can be solved only for $y_1 = 0$. By substituing $y_1 = 0$ in eq.(5) we get

$$[0, y_2]\left[\begin{bmatrix} B^{-1} \\ 0 \end{bmatrix}, \begin{bmatrix} B^{-1}(I-A)B^{-1} \\ -DB^{-1} \end{bmatrix}, \begin{bmatrix} (B^{-1}(I-A))^2 B^{-1} \\ -DB^{-1}(I-A)B^{-1} \end{bmatrix}, \ldots \begin{bmatrix} (B^{-1}(I-A))^{m+n-1} B^{-1} \\ -D(B^{-1}(I-A))^{m+n-2} B^{-1} \end{bmatrix}\right] = 0,$$

after multiplication we obtain

$$\left[0, -y_2 DB^{-1}, -y_2 DB^{-1}(I-A)B^{-1}, \ldots, -y_2 D(B^{-1}(I-A))^{m+n-2} B^{-1}\right] = 0.$$

Taking the common factor $y_2 D$ out of the left-hand side of the previous equation we have

$$y_2 D\left[0, -B^{-1}, -B^{-1}(I-A)B^{-1}, \ldots, -(B^{-1}(I-A))^{m+n-2} B^{-1}\right] = 0. \quad (6)$$

Supposing that *rank D* = *m* then the above equality holds only for $y_2 = 0$, hence *rankW* = *m*+*n*.

Conversely, suppose *rankW* = *m*+*n* we prove that *rank D* = *m*. If *rank D* < *m* then there exists $y_2 \neq 0$ such that $y_2 D = 0$, that is the equality (6) holds not only for $y_2 = 0$ therefore *rankW*<*m*+*n*. This completes the proof of the Lemma.

In the next section supposing a given control, that is supposing a given consumption we examine the trajectories of system (1) and we examine the effect of a balanced path on the use of the non-renewable environmental resources.

4. Balanced growth path

In the following we study the balanced growth solution of the system (1) corresponding to a given growth rate α ($\alpha \geq 0$) supposing that both production and consumption increase at the same rate α. Under the assumptions given the balanced growth solution of the model (1) have the form $x_t = (1+\alpha)^t x_0$ and $c_t = (1+\alpha)^t c_0$ where $\alpha \geq 0$ **.** Let us substitute the former expressions for x_t and c_t in the eq. (1). It follows that the following must hold

$$(I - A - \alpha B)\, x_0 = c_0 \quad (7)$$

From the eq. (7) follows that the output configuration x_0 corresponding to the balanced growth path, depends on α and c_0 as well. Next we make conditions for the existence of nonnegative output configuration x_0.

Throughout we will use the following notation. Let *M* be an arbitrary ($n \times n$) matrix. Then *M* ≥ 0 (*M* nonnegative) means each element of *M* is nonnegative and for nonnegative *M*, $\lambda_1(M)$ denotes the Frobenius root of *M* that is the nonnegative real eigenvalue of *M* dominant in modulus.

Lemma *2.*
The output configuration x_0 corresponding to eq. (7) exists and it is nonnegative if $\alpha\in[0, \alpha_0[$, where α_0 is the marginal growth rate such that $\lambda_1(A + \alpha_0 B) = 1$.

Proof. Since matrix $A + \alpha B$ is nonnegative according to Perron-Frobenius's theorem (Morishima 1964), there exists the Frobenius root of matrix $A + \alpha B$. If this latter is less than 1 then the inverse of matrix $I-A-\alpha B$ exists and it is nonnegative too.

The condition $\lambda_1(A + \alpha B) <1$ it is obviously fulfilled for $\alpha = 0$ in case of productive systems. By applying Perron-Frobenius's theorem it is easy to see that on the one hand $\lambda_1(A + \alpha B)$ is a monotone increasing function of α for $\alpha\geq 0$ and on the other hand there exists such an α^*, ($\alpha^* = \frac{1}{\lambda_1(B)}$) such that $\lambda_1(A + \alpha^* B) \geq 1$. From the previous reasoning and because $\lambda_1(A + \alpha B)$ is a continuous function of α, it follows that by increasing the value of α the value of $\lambda_1(A + \alpha B)$ will increase too. Hence there exists an α_0 such that $\lambda_1(A + \alpha_0 B) = 1$. Then we conclude that for every α chosen from the interval of $[0, \alpha_0[$ holds the inequality $\lambda_1(A + \alpha B) <1$ which implies that $(I-A-\alpha B)^{-1}$ exists and $(I-A-\alpha B)^{-1}\geq 0$. Therefore for every α, $\alpha\in[0, \alpha_0 [$ and given consumption vector c_0 there exists a nonnegative output configuration x_0, that is

$$x_0 (\alpha) = (I-A-\alpha B)^{-1} c_0 . \qquad (8)$$

Proposition 1. The marginal growth rate α_0 is the solution of the following eigenvalue problem of

$$\mathbf{B}^{-1}(\mathbf{E}-\mathbf{A})\mathbf{v} = \alpha_0 \mathbf{v}$$

***for some nonzero* n*-dimensional vector* v.**

Proof. According to Lemma 2. for the marginal growth rate α_0 holds the equality $\lambda_1(A + \alpha_0 B) = 1$. This implies that there exists a nonzero vector v such that $(A + \alpha_0 B)v = v$. Equivalently, this relation may be written as $(E-A)^{-1}Bv = \frac{1}{\alpha_0}$v which is equivalent to the eigenvalue problem which was to be proved.

***Proposition* 2.** Let C and D be $(n\times n)$ nonnegative matrices and $C \leq D$. Then holds $C^2 \leq D^2$.

Proof. The assumption means that $0\leq c_{ij} \leq d_{ij}$ for all $i,j \in \overline{1,n}$. Using the former inequalities and the rule for multiplication of matrices we obtain $C^2 = \left[\sum_{k=1}^{n} c_{ik}c_{kj}\right] \leq \left[\sum_{k=1}^{n} d_{ik}d_{kj}\right] = D^2$.

Remark. It is easy to see that the proposition 1. holds also for any power of matrices C and D .

***Lemma* 3.**

Consider that the nonnegative output configuration $x_0(\alpha)$ exists. Then

(i) $x_0(\alpha)$ is a monotone increasing function of α, for $\alpha\in[0, \alpha_0[$

(ii) $x_0(\alpha)$ is an unbounded function for $\alpha\in[0, \alpha_0[$.

Proof. (i) Let α_1, $\alpha_2\in[0, \alpha_0[$ and $\alpha_1\leq \alpha_2$. Since $\alpha_1\in[0, \alpha_0[$ therefore $\lambda_1(A + \alpha_1 B) <1$. The latter implies that $(I-A-\alpha_1 B)^{-1}$ exists and $(I-A-\alpha_1 B)^{-1} = \sum_{i=0}^{\infty}(A+\alpha_1 B)^i$. We get in similar way that $(I-A-\alpha_2 B)^{-1} = \sum_{i=0}^{\infty}(A+\alpha_2 B)^i$. By applying the Proposition 1. and the Remark for $C = A + \alpha_1 B$ and $D = A + \alpha_2 B$

we obtain $\sum_{i=0}^{\infty}(A+\alpha_1 B)^i \leq \sum_{i=0}^{\infty}(A+\alpha_2 B)^i$ that is $(I-A-\alpha_1 B)^{-1} \leq (I-A-\alpha_2 B)^{-1}$. By multiplying both sides of the former inequality by the nonnegative vector c_0 we get that $x_0(\alpha_1) \leq x_0(\alpha_2)$. This completes the proof of (i) of the lemma.

(ii) Assume that $\alpha \to \alpha_0$. Using the continuity of $\lambda_1(A + \alpha B)$ we have $\lambda_1(A + \alpha B) \to 1$. It is easy to see that $\lambda_1 (E-A-\alpha B)^{-1} = \frac{1}{1-\lambda_1(A+\alpha B)}$. Hence for $\alpha \to \alpha_0$ we have $\lambda_1 (I-A-\alpha B)^{-1} \to \infty$. Then if the nonnegative eigenvector corresponding to $\lambda_1 (I-A-\alpha B)^{-1}$ is not normal to the nonnegative consumption vector c_0 we have $x_0(\alpha)\to \infty$. This concludes that $x_0(\alpha)$ increases over every boundary.

5. The examination of the resources

In the following we investgate the evaluation of the stocks of exhaustible resources corresponding to the balanced growth path of the system (3).We first determine the solution of the difference eq.(2) for a balanced growth with a nonnegative growth rate α for the output as well as the final consumption. Let us substitute the balanced growth solution $x_t = (1+\alpha)^t x_0$ of eq.(1) in eq.(2) then by a simple iteration we obtain the following formula (9) for the stocks of resources

$$R_t = R_0 - \frac{(1+\alpha)^t - 1}{\alpha} D x_0(\alpha). \tag{9}$$

Next we give the following lemmas:

***Lemma* 4.**

The stocks of resources R_t in dependence of α is a monotone decreasing function of α, for $\alpha\in[0, \alpha_0[$.

The proof of this lemma is fairly simple on the basis of Lemma 3. and by using that $(1+\alpha)^t$ is a monotone increasing geometric sequence for $1+\alpha \geq 1$.

***Lemma* 5.**

The stocks of resources R_t in dependence of c_0 is a monotone decreasing function of c_0 for any nonnegative consumption vector c_0.

This statement can be prooved by similar arguments as that in the proof of Lemma 4.

Remark. In economic terms these lemmas mean that a lower rate of growth or lower consumption level results in a longer lasting of the stocks of non-renewable resources.

6. Estimation of the exhaustion of resources

In the below evaluation we estimate how long will cover the stocks of scare resources the inputs of production under the assumption we have already made related to the growth of production and consumption, as well. In the expression (9) we intend to determine the maximal t for which the stocks of resources R_t are still nonnegative ($R_t \geq 0$). Denote by T the requested maximal time period. Then it must holds the following equality for this T

$$\frac{(1+\alpha)^T - 1}{\alpha} = \min_i \left(\frac{(R_0)_i}{(Dx_0(\alpha))_i} \right)$$

where $(\cdot)_i$ denotes the i^{th} component of the respective vector. By expressing T we obtain

$$T = \frac{\ln\left(\alpha \min\limits_i \frac{(R_0)_i}{(Dx_0(\alpha))_i} + 1 \right)}{\ln(1+\alpha)} \qquad \text{for } \alpha \in]0, \alpha_0[.$$

7. Conclusions and further research

In this paper we have investigated a generalised dynamic Leontief model. The basic model was extended with the resource exploitation. It was proved that this augmented model is controllable if the input matrix of resources has a full rank. In the second part of the paper we have examined the effect of the balanced growth path on the resource exploitation. It was shown that the stock of non-renewable resources lasts longer if both the growth rate and the consumption level are chosen lower.

In a next research we could examine the effect of the balanced growth path on the quality of the renewable environmental resources. How will influence the use of the environmental resources (emission) this balanced path if the emission level is limited by a legal environmental standard. Another possibility would be an extension of the analysed model with the recycling of reusable materials. In this case the coefficients of input matrix of resources will be smaller than it were before and the environmental resources last longer for the next generation. This investigation will supply the concept of the sustainable development.

These examinations ignore the influence of prices on the growth path. This dual problem could reply on the question about the economic effectivity of production. This newly established price system could control the production in an environmental consciousness way.

References

Bródy, A. (1980),Ciklus és szabályozás,Közgazdasági és Jogi Könyvkiadó, Budapest
Elaydi, S. N.(1996), An Introduction to Difference Equations, Springer
Hirsch, M.W. and S. Smale (1974), Differential Equations, Dynamical Systems, and Linear Algebra, Academic Press,Inc., p.130
Kalman, R.E.(1960),Contributions to the Theory of Optimal Control, Bol. Socied. Mat.Mexicana
Leontief, W. (1986), Input-Output Economics, Oxford University Press,
Morishima, M.(1964), Equilibrium Stability and Growth, Oxford University Press, p. 195
Schoonbeek, L. (1990), The Size of the Balanced Growth Rate in the Dynamic Leontief Model, Economic Systems Research, Vol. 2, No. 4, 345-349
Zalai, E. (2000),Matematikai közgazdaságtan,KJK-KERSZÖV, Budapest

Authors:

Adél Floriska[1], Imre Dobos[2]
[1] Institute Mathematics and Informatics, Szent István University, H-2100 Gödöllő, Páter Károly u.1, Hungary
E-mail:afloriska@mszi.gau.hu
[2] Department of Business Economics, Budapest University of Economics and Public Administration, H-1053 Budapest, Veres Pálné u. 36, Hungary, imre.dobos@bkae.hu